Growth and inequality examined by integrating the Walrasian general equilibrium and neoclassical growth theories

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Abstract

This paper builds a heterogeneous-households growth model of a small open economy with fixed resource (land) by integrating the Walrasian general equilibrium and neoclassical growth theories. The production side consists of two sectors. We use an alternative utility function proposed by Zhang, which enable us to develop a dynamic growth model with genuine heterogeneity. The wealth and income inequality is due to household heterogeneity in preferences and human capital as well as the households' initial wealth. This is different from the standard Ramsey-type heterogeneous-households growth models, for instance, by Turnovsky and Garcia-Penalosa (2008), where agents are heterogeneous only in their initial capital endowment, not in preference or/and human capital. We simulate the model for an economy with three types of households. The system has a unique stable equilibrium point. We also simulate the motion of the national economy and carry out comparative dynamic analysis with regard to changes in the rate of interest, the population, the propensity to stay at home, and the propensity to save. The comparative dynamic analysis provides some important insights.

Keywords: growth, inequality, capital accumulation, small open economy, wealth and income distribution

JEL Classification: O31, E31

1. Introduction

Growth and inequality in wealth and income distribution has caused a lot of attention in economic growth theory (e.g., Alesina and Rodrik, 1994; Persson and Tabellini, 1994; Perotti, 1996; Li and Zou, 1998; Forbes, 2000; Barro, 2000; Chen and Turnovsky, 2010; Zhang, 2012b). Issues about growth and inequality are often main concerns in public debates in different parts of the world. It has become clear that it is necessary to study growth and inequality in wealth and income distribution in a general equilibrium growth framework. The purpose of his study is to examine growth and inequality in an integrated framework

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of the Walrasian general equilibrium and neoclassical growth theory. The Walrasian general equilibrium theory deals with analyzing equilibrium of economic exchanges with heterogeneous households and multiple sectors, while the neoclassical growth theory is focused on wealth accumulation.

The Walrasian general equilibrium theory of pure exchange and production economies has been the mainstream in economic theories with microeconomic foundation. It is proposed by Walras and further developed and refined by Arrow, Debreu and others in the 1950s (e.g., Walras, 1874; Arrow and Debreu, 1954; Gale, 1955; Nikaido, 1956, 1968; Debreu, 1959; McKenzie, 1959; Arrow and Hahn, 1971; Arrow, 1974; Mas-Colell et al., 1995). The theory is mainly concerned with market equilibrium with economic mechanisms of production, consumption, and exchanges with heterogeneous industries and households. The model in our study is Walrasian in the sense that for given levels of wealth there are competitive market equilibriums with heterogeneous industries and households. Our model is also based on neoclassical growth theory. It is well-known that the Walrasian general theory fails to be generalized and extended to growth theory of heterogeneous households with endogenous wealth. Although Walras introduced saving and capital accumulation in his general equilibrium theory, he did not succeed in taking account of capital accumulation in the general equilibrium theory (e.g., Impicciatore et al., 2012). Different attempts have been made to introduce capital accumulation into Walras' framework of heterogeneous households (e.g., Morishima, 1964, 1977; Diewert, 1977; Eatwell, 1987; Dana et al., 1989; Jensen and Larsen, 2005; Montesano, 2008). All these models lack proper microeconomic foundation for wealth accumulation of heterogeneous households. On the other hand, neoclassical growth theory has been developed since the 1950s (e.g., Solow, 1956; Burmeister and Dobell, 1970; Barro and Sala-i-Martin, 1995). The theory models endogenous wealth accumulation with microeconomic foundation (e.g., Ramsey model). This study follows Uzawa's two sector growth model in describing capital accumulation and economic structure (Uzawa, 1961). Uzawa's two-sector model has been generalized and extended in different ways over years (see, Diamond, 1965; Stiglitz, 1967; Mino, 1996; and Drugeon and Venditti, 2001; Jensen, 2003; and Jensen, 2005). Nevertheless, only a few attempts have been done to examine inequality in income and inequality within the Uzawa two-sector model. We will basically follow the Uzawa model in modelling economic structural change. Our approach is influenced not only by the neoclassical growth theory but also by the post-Keynesian theory of growth and distribution (e.g., Pasinetti, 1974; Salvadori, 1991). In most of the Post-Keynesian growth models with heterogeneous classes economic systems have a single production sector. One exception is by Stiglitz (1967) who proposes a growth model of two sectors and two classes. The Stigliz model synthesizes the post-Keynesian theory and Uzawa's two-sector model. But there are few further studies along the research line. This study deals with similar economic issues to those addressed by the Stigliz model but in an alternative approach to household behavior. Moreover, labor supply is an endogenous variable and economy is open in our approach, while labor supply is fixed and the economy is closed in the Stiglitz model.

This study deals with growth and inequality by integrating the neoclassical growth theory with the general equilibrium theory. Moreover, our model is designed for an open economy. As observed by Chen and Turnovsky (2010, p. 332), "virtually the entire growth-inequality literature is restricted to a closed economy, which is a severe shortcoming given the increasing openness characterizing most economies". As most of economies in the world are open, it is significant to examine effects of international markets on open national economies. This study deals with dynamics of an open small economy. There are many studies on economic growth of small open economies (e.g., Obstfeld and Rogoff, 1998; Gali and Monacelli, 2005), even though most of these theoretical growth models are built for open small economies with homogeneous population. The purpose of this paper is to build a two-sector heterogeneous-households growth model for a small open economy to examine dynamics of wealth and income distribution with capital accumulation as the main engine of economic growth.

Chen and Turnovsky (2010, pp. 331-332) recognized that "Because an economy's growth rate and its income distribution are both endogenous equilibrium outcomes of the economic system, the income inequality-growth relationship – whether positive or negative – will reflect the underlying set of forces to which both are reacting. To understand these linkages, it is necessary to adopt a structural, consistently-specified general equilibrium approach." Yet, it is argued that economics still needs an analytical framework for properly dealing with issues related to income and wealth distribution and economic growth with microeconomic foundation. It is not easy to model economic growth with heterogeneous households (Sorger, 2000). The reason is summarized by Chen and Turnovsky (2010, p. 332) as follows: "In a completely general setup, where the equilibrium growth rate and income distribution are mutually dependent, their joint determination and the analysis of their relationship becomes intractable". Hence, irrespective of many efforts over years by theoretical economists, issues related to growth and inequality remain unsolved in a general dynamic equilibrium approach. This study develops a general equilibrium growth model by combining the economic mechanisms of the Walrasian general equilibrium and neoclassical growth theories with an alternative approach to households proposed by Zhang (1993). This approach helps us to overcome the problem of "analysis of their relationship becomes intractable". The paper synthesizes the ideas in the economic growth model for an open-small economy by Zhang (2012a) and the one-sector neoclassical growth model with heterogeneous households by Zhang (2012b). By combing the basic structures of the two models, we build a framework for unifying the Walrasian general equilibrium theory and the neoclassical growth theory. This study is similar to the growth model for an open economy and elastic labor supply with heterogeneous households proposed by Chen and Turnovsky (2010). Like this study, their model is also concerned with issues related to the growth and inequality relations for a small open economy. The main difference is that this study uses an alternative utility function proposed by Zhang, while Chen and Turnovsky use the traditional Ramsey approach to deal with household behavior. As mentioned later on, the Ramsey approach implies that they have to assume that heterogeneous agents differ only in their initial endowments of wealth, while the model based on Zhang's approach allows us not only allow different agents to have different initial endowments of wealth, but also allow heterogeneous households to differ in preferences. The remainder of this study is organized as follows. Section 2 defines the model. Section 3 examines dynamic properties of the model. Section 4 carries out comparative dynamic analysis with regard to some parameters. The appendix gives the procedure for determining the monotonous differential equations in section 3.

2. The Growth Model of Economic Structure and Heterogeneous Households

We now build a small open growth model with heterogeneous households and multiple sectors. The small-open economy produces two goods: an internationally traded good (called industrial good) and a non-traded good (called services). The classification of the economic sectors is similar to that in a growth model of a small open economy, for instance, by Brock (1988), in which goods and services are divided into traded and non-traded. It should be noted that the core model in the neoclassical growth theory was the Solow one-sector growth model (Solow, 1956). The one-sector model is not suitable for analyzing economic structural change and price changes of various goods, initial extensions of the Solow model to multiple sectors were initially proposed by Uzawa (1961, 1963), Meade (1961) and Kurz (1963). In the traditional two-sector economy, output of the capital sector is used entirely for investment and that of the consumption sector for consumption. Economists have made many efforts in generalizing and extending the Uzawa two-sector model by, for instance, introducing more general production functions, more sectors, money, externalities, knowledge, human capital, and fictions in different markets (for instance, Takayama, 1985; Galor, 1992; Azariadis, 1993; Harrison, 2003; Jensen, 2003; Cremers, 2006; Herrendorf and Valentinyi, 2006; Li and Lin, 2008; Stockman, 2009; Jensen and Lehmijoki, 2011). This study adapts the traditional two-sector economic structure to an open economy. It should be noted that Jensen et al. (2001) develop a framework for analyzing the dynamics of small open economies with CES sector technologies. Although the economic production structure of our model is similar to this model, the model by Jensen et al. is developed with the homogeneous population and the traditional approach to the behavior of households. An open economy can import goods and services and borrow resources from the rest of the world or exports goods and lend resources abroad. There is a single internationally tradable good, called industrial good, in the world economy and the price of the industrial good is unity fixed in global markets. Capital depreciates at a constant exponential rate, δ_k, which is independent of the manner of use. We assume that the economy is too small to affect the world rate of interest, r^*. The households hold wealth and land and receive income from wages, land rent, and interest payments of wealth. Land is only for residential and service use. Technologies of the production sectors are characterized of constant returns to scale. All markets are perfectly competitive and capital and labor are completely mobile between the two sectors. Capital is perfectly mobile in international market and we neglect possibility of emigration or/and immigration. We assume that labor is homogeneous and is fixed.

The population is classified into J groups, each group with fixed population, $\bar{N}_j$.

Let $T_j(t)$ stand for the work time of a representative household of group j and $N(t)$ for the flow of labor services used at time t for production. We assume that labor is always fully employed. We have

$$N(t) = \sum_{j=1}^{J} h_j T_j(t) \bar{N}_j, \tag{1}$$

where h_j are the levels of human capital of group j.

Industrial sector

The industrial sector uses capital and labor as inputs. We use subscript index, i and s, to denote respectively the industrial and service sectors. Let $K_j(t)$ and $N_j(t)$ stand for the capital stocks and labor force employed by sector j, $j = i, s$, at time t. We use $F_j(t)$ to represent the output level of sector j. The production function of the industrial sector is

$$F_i(t) = A_i K_i^{\alpha_i}(t) N_i^{\beta_i}(t), \ \alpha_i, \beta_i > 0, \ \alpha_i + \beta_i = 1, \tag{2}$$

where A_i, α_i, and β_i are parameters. Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. The wage rate, $w(t)$, is determined in domestic market. Hence, for any individual firm, r^* and $w(t)$ are given at any point in time. The industrial sector chooses $K_i(t)$ and $N_i(t)$ to maximize profits. The marginal conditions are

$$r^* + \delta_k = \alpha_i A_i k_i^{-\beta_i}(t), \ w(t) = \beta_i A_i k_i^{\alpha_i}(t), \tag{3}$$

where $k_i(t) \equiv K_i(t) / N_i(t)$. As r^* is fixed, from (3) we have

$$k_i = \left(\frac{\alpha_i A_i}{r^* + \delta_k} \right)^{1/\beta_i}, \ w = \beta_i A_i k_i^{\alpha_i}. \tag{4}$$

Hence, we can treat k_i and w as functions of r^* and A_i.

Service sector

The service sector employs three inputs, capital $K_s(t)$, labor force $N_s(t)$, and land $L_s(t)$, to produce services. We specify the production function as

$$F_s(t) = A_s K_s^{\alpha_s}(t) N_s^{\beta_s}(t) L_s^{\gamma_s}(t), \ \alpha_s, \beta_s, \gamma_s > 0, \ \alpha_s + \beta_s + \gamma_s = 1, \tag{5}$$

where A_s, α_s, β_s, and γ_s are parameters. We use $p(t)$ and $R(t)$ to represent respectively the price of services and the land rent. The marginal conditions are

$$r^* + \delta_k = \alpha_s A_s\, p(t) k_s^{\alpha_s - 1}(t) l_s^{\gamma_s}(t), \;\; w = \beta_s A_s\, p(t) k_s^{\alpha_s}(t) l_s^{\gamma_s}(t),$$
$$R(t) = \gamma_s A_s\, p(t) k_s^{\alpha_s}(t) l_s^{\gamma_s - 1}(t), \tag{6}$$

where

$$k_s(t) \equiv \frac{K_s(t)}{N_s(t)}, \;\; l_s(t) \equiv \frac{L_s(t)}{N_s(t)}.$$

Equations (6) imply

$$k_s = \frac{\alpha_s w}{\beta_s r_\delta}. \tag{7}$$

Hence, we can treat k_s as a function of r^* and A_i.

Full employment of capital and labor

The total capital stocks utilized by the small-open economy, $K(t)$, is distributed between the two sectors. The capital stock utilized by the economy is not necessary to be owned by domestic residents. Full employment of labor and capital implies

$$K_i(t) + K_s(t) = K(t), \;\;\; N_i(t) + N_s(t) = N(t). \tag{8}$$

Equations (8) imply

$$k_i N_i(t) + k_s N_s(t) = K(t), \;\;\;\; N_i(t) + N_s(t) = N(t). \tag{9}$$

In (9), k_i and k_s, are uniquely determined by the rate of interest which is fixed in international market. Solve (8)

$$N_i(t) = \left(K(t) - k_s N(t)\right) k_v, \;\; N_s(t) = \left(k_i N(t) - K(t)\right) k_v, \tag{10}$$

where $k_v \equiv (k_i - k_s)^{-1}$. We require $k_i \neq k_s$. The labor distribution is uniquely determined by the total capital utilized by the economy.

Behavior of households

We use L and $R(t)$ to stand for the fixed land and land rent, respectively. The representative household obtains income from land ownership, wealth and wage. To decide income, we need to determine who owns the land and how the land rent is distributed. Land may be owned by different agents under different institutions. For instance, in the literature of urban economics two types of land distribution are often assumed. The one is the so-called absentee landlord. Under this assumption the landlords spend their land incomes outside the economic system. The another type, for instance as accepted in Kanemoto (1980), assumes that the urban government rents the land from the landowners at certain rent and sublets it to households at the market rent, using the net revenue to subsidize city residents equally. This study assumes the land equally owned by the population, which implies that the households equally share the land rent income. The national land rent income is equal to $L\,R(t)$. The land rent income per household $\bar{r}(t)$ is

$$\bar{r}(t) = \frac{L\,R(t)}{\bar{N}}, \tag{11}$$

where $\bar{N}$ is the total population

$$\bar{N} = \sum_{j=1}^{J} \bar{N}_j .$$

Households choose lot size, consumption levels of industrial goods and services, and save. We use $\bar{k}_j(t)$ to stand for wealth per capita owned by household j. The current income is

$$y_j(t) = r^* \bar{k}_j(t) + h_j\, w T_j(t) + \bar{r}(t), \tag{12}$$

where $r^* \bar{k}_j$ is the interest income, $h_j\, w T_j(t)$ the wage income, and $\bar{r}(t)$ the land rent income. We call $y(t)$ the current income in the sense that it comes from consumers' wages and current earnings from ownership of wealth. In the Solow one-sector growth model it is assumed that a fixed proportion of the current income is saved for the future consumption. Nevertheless, the Solowian approach neglects possible effects of wealth on households. Moreover, the available expenditure that a household spends is not necessary less than the current income as assumed in the Solow model. When the current income is not sufficient for consuming, the household may spend the past saving. We note that the total value of the wealth of household is $p_i(t)\bar{k}_j(t)$, with $p_i(t) = 1$ at any t, where $p_i(t)$ is the price of the industrial good. It is assumed that selling and buying wealth can be conducted instantaneously without any transaction cost. The disposable income is the current income plus the value of the household's wealth

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t) . \tag{13}$$

The disposable income is used for saving and consumption. Let $T_{hj}(t)$ stand for the leisure time at time t and T_0 the (fixed) available time for work and leisure. The time is distributed between leisure and work

$$T_j(t) + T_{hj}(t) = T_0 . \tag{14}$$

The household spends the disposable income on the lot size, consumption of services, consumption of industrial goods, and saving. The budget constraint is

$$R_j(t)l_j(t) + p(t)c_{sj}(t) + c_{ij}(t) + s_j(t) = \hat{y}_j(t). \tag{15}$$

This equation implies that the household's disposable income is entirely distributed between the consumption and saving. Inserting (14) and (13) in (15) implies

$$w_j T_{hj}(t) + R_j(t)l_j(t) + p(t)c_{sj}(t) + c_{ij}(t) + s_j(t) = \bar{y}_j(t), \tag{16}$$

in which $w_j \equiv h_j w$ and

$$\bar{y}_j(t) \equiv (1 + r\ )\bar{k}_j(t) + w_j T\ + \bar{r}(t). \tag{17}$$

The utility function, $U_j(t)$, of the household is dependent on $T_{hj}(t)$, $l_j(t)$, $c_{sj}(t)$, $c_{ij}(t)$ and $s_j(t)$ as follows

$$U_j(t) = \theta_j T_{hj}^{\sigma_{0j}}(t) l_j^{\eta_{0j}}(t) c_{sj}^{\gamma_{0j}}(t) c_{ij}^{\xi_{0j}}(t) s_j^{\lambda_{0j}}, \ \ \sigma_{0j}, \eta_{0j}, \gamma_{0j}, \xi_{0j}, \lambda_{0j} > 0,$$

in which σ_{0j}, η_{0j}, γ_{0j}, ξ_{0j}, and λ_{0j} are a typical household's utility elasticity of leisure time, lot size, services, industrial goods, and saving. We call σ_{0j}, η_{0j}, γ_{0j}, ξ_{0j}, and λ_{0j} household $j's$ propensities to leisure time, to consume housing, to consume services, to consume industrial goods, and to hold wealth, respectively.

It should be noted that there are some other studies which deal with similar issues like in this study. The main difference of the traditional approach from this study is about behavior of households. To illustrate the difference, we mention a study by Turnovsky and Garcia-Penalosa (2008). Their model also studies the dynamics of the distributions of wealth and income. But their model is developed in in a Ramsey model in which agents differ in their initial capital endowment and where the labor supply is endogenous. Their model assumes that the agent maximizes lifetime utility, which is a function of both consumption and the amount of leisure time as

$$Max \int_0^\infty T_{hj}^{\sigma}(t) c_{ij}^{\xi}(t) e^{-\beta t}\, dt.$$

The preference parameters σ, ξ, and β are the same for all types of the households. Accordingly, the so-called heterogeneous households in this approach are not heterogeneous in preference, but are different only in initial wealth. The identical preference among different types of households is "necessary" because of a well-known property of the Ramsey-type growth theory as described by Turnovsky and Garcia-Penalosa (2008), "Early work examining the evolution of the distribution of wealth in the Ramsey model assumed agents that differ in their rate of time preferences. In this framework, the most patient agent ends up holding all the capital in the long run…". This implies that if households are different in their time preferences, the entire wealth is help only by one household and the rest of the population has no wealth in the long term. Obviously, this type of models with different time preferences is not interesting. Nevertheless, real world requires that a useful growth theory deals with heterogeneous households with different preferences. The Ramsey-type growth may not be suitable for exploring variety and heterogeneity in economic systems. Zhang (1993) proposes an alternative approach to household behavior to analyze heterogeneous households. This study is another application of Zhang's idea for modeling behavior of households.

Maximizing $U_j(t)$ subject to the budget constraint (17) implies

$$T_{hj}(t) = \frac{\sigma_j \bar{y}_j(t)}{w_j}, \; l_j(t) = \frac{\eta_j \bar{y}_j(t)}{R(t)}, \; c_{sj}(t) = \frac{\gamma_j \bar{y}_j(t)}{p(t)}, \; c_{ij}(t) = \xi_j \bar{y}_j(t), \; s_j(t) = \lambda_j \bar{y}_j(t), \tag{18}$$

where

$$\sigma_j \equiv \rho_j \sigma_{0j}, \; \gamma_j \equiv \rho_j \gamma_{0j}, \; \xi_j \equiv \rho_j \xi_{0j}, \; \lambda_j \equiv \rho_j \lambda_{0j},$$

$$\rho_j \equiv \frac{1}{\sigma_{0j} + \eta_{0j} + \gamma_{0j} + \xi_{0j} + \lambda_{0j}}.$$

According to the definition of $s_j(t)$, the change in wealth per capita of household j is

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t). \tag{19}$$

This equation simply implies that the change in wealth is the saving minus dissaving.

Full use of land and demand of and supply for services

Land is used for the residential use and service production

$$\sum_{j=1}^{J} l_j(t)\bar{N}_j + L_s(t) = L. \tag{20}$$

The equilibrium condition for services is

$$\sum_{j=1}^{J} c_{sj}(t)\bar{N}_j = F_s(t). \tag{21}$$

The national wealth is equal to the sum of the wealth owned by all the households in the country

$$\bar{K}(t) = \sum_{j=1}^{J} \bar{k}_j(t)\bar{N}_j. \tag{22}$$

The current return from net asset

As $\bar{K}(t)$ is the wealth owned by the population and $K(t)$ is the capital stock employed by the country, $\bar{K}(t) - K(t)$ is net asset in trade balance. We use $E(t)$ to denote the current return from net asset, that is

$$E(t) = r^* \left(\bar{K}(t) - K(t)\right). \tag{23}$$

We have thus built the model. It can be seen that the model is structurally a unification of the Walrasian general equilibrium and neoclassical growth theory. If we neglect the wealth accumulation and capital depreciation (i.e., capital being constant), then the model with heterogeneous households and multiple sectors belongs to the Walrasian general equilibrium theory. If we allow the households to be homogeneous, then the model is similar to the Uzawa model in the neoclassical growth theory. It should be noted that our model is not identical to the neoclassical growth theory. A main deviation from the traditional neoclassical growth theory is how to model behavior of households. We use an alternative utility function proposed by Zhang.

3. The Dynamics of the Economy

The model has many variables and these variables are interrelated to each other in complicated ways. As there are different types of households and households have different propensities and human capital levels, the dynamics should be nonlinear and of high dimension. It is difficult to get analytical properties of the nonlinear differential equations. Nevertheless, we show that we can plot the motion of the system with initial conditions with computer. Before stating the lemma, we note that we have determined k_i, w, w_j, and k_s as functions of r^*, h_j and A_i. This implies that these variables are exogenously determined by the domestic technology, the human capital levels, and the global goods and capital markets. In the rest of this study we consider these variables constant. The following lemma gives a computational procedure for plotting the motion of the dynamic system.

Lemma

The motion of the economic system with J types of household is governed the following J nonlinear differential equations

$$\dot{R}(t) = \Omega_1\left(R(t), \left\{\bar{k}_j(t)\right\}\right),$$

$$\dot{\bar{k}}_j(t) = \Omega_j\left(R(t), \left\{\bar{k}_j\right\}(t)\right), \quad j = 2, ..., J, \tag{24}$$

where Ω_j are functions of $R(t)$ and $\left\{\bar{k}_j(t)\right\} = \left(\bar{k}_2(t), ..., \bar{k}_J(t)\right)$ given in the Appendix.

The lemma confirms that we have a set of nonlinear differential equations from which we can explicitly determine the motion of the J variables, $R(t)$ and $(\bar{k}_2(t), ..., \bar{k}_J(t))$. The dimension of the dynamic system is equal to the number of types of household. In a Walrasian general equilibrium theory where households are different from each other, the dimension of the dynamic system is the same as the population. As shown in the Appendix, we use $R(t)$ rather than $\bar{k}_1(t)$ in the dynamic analysis as this enables us to find the set of differential equations by which we can solve the motion of all the variables by simulation. As shown in the Appendix, once we determine the values of $R(t)$ and $\{\bar{k}_j(t)\}$ at any point in time by the equations in the lemma, then we can obtain the values of all the other variables as functions of $R(t)$ and $\{\bar{k}_j(t)\}$ by the following procedure: $\bar{k}_1(t)$ by (A13) → $\bar{y}(t)$ by (A4) → $p(t)$ by (A10) → $T_j(t)$ by (A15) → $T_{hj}(t)$, $l_j(t)$, $c_{ij}(t)$, $c_{sj}(t)$, $s_j(t)$ by (18) → $N(t)$ by (A16) → $K(t)$ by (A12) → $K_i(t)$ and $K_s(t)$ by (A1) → $N_i(t)$ and $N_s(t)$ by (9) → $D_T(t)$ by (10) → $\bar{K}(t)$ by (22) → $L_s(t)$ by (A2) → $F_i(t)$ by (2) → $F_s(t)$ by (5).

The lemma and this computational procedure allow us to plot the motion of the economic system once we know the initial conditions of the system and the rate of interest in the global market. It should be noted that from the proving process of the lemma in the appendix, it is straightforward to see that we still can simulate the motion of the system with the rate of interest as a function of time. Following the procedure with portable computer, we can illustrate the motion of the system. For simulation, we choose $J = 3$ and specify the parameter values

$$\bar{N}_1 = 2,\ \bar{N}_2 = 3,\ \bar{N}_3 = 5,\ h_1 = 3,\ h_2 = 1,\ h_3 = 0.5,\ A_i = 1.5,\ A_s = 1,\ \alpha_i = 0.3,$$
$$\alpha_s = 0.3,\ \beta_s = 0.6,\ r = 0.06,\ T_0 = 1,\ L = 10,\ \lambda_{01} = 0.8,\ \xi_{01} = 0.15,\ \gamma_{01} = 0.06,$$
$$\eta_{01} = 0.08,\ \sigma_{01} = 0.2,\ \lambda_{02} = 0.7,\ \xi_{02} = 0.15,\ \gamma_{02} = 0.07,\ \eta_{02} = 0.06,\ \sigma_{02} = 0.22,$$
$$\lambda_{03} = 0.65,\ \xi_{03} = 0.18,\ \gamma_{03} = 0.08,\ \eta_{03} = 0.05,\ \sigma_{03} = 0.25,\ \delta_k = 0.05. \quad (25)$$

The rate of interest is fixed at 6 per cent. Group $1's$, $2's$, and $3's$ population are respectively 2, 3, and 5. Group $1's$, $2's$, and $3's$ level of human capital are respectively 3, 1, and 0.5. Group 1 (3) has the smallest (largest) population size and highest (lowest) human capital. The groups have also different preferences. The total available time is unity and the land is 10. From the previous section we know that capital intensities and wage rates are determined by the international rate of interest and the domestic technology. From (25) we calculate

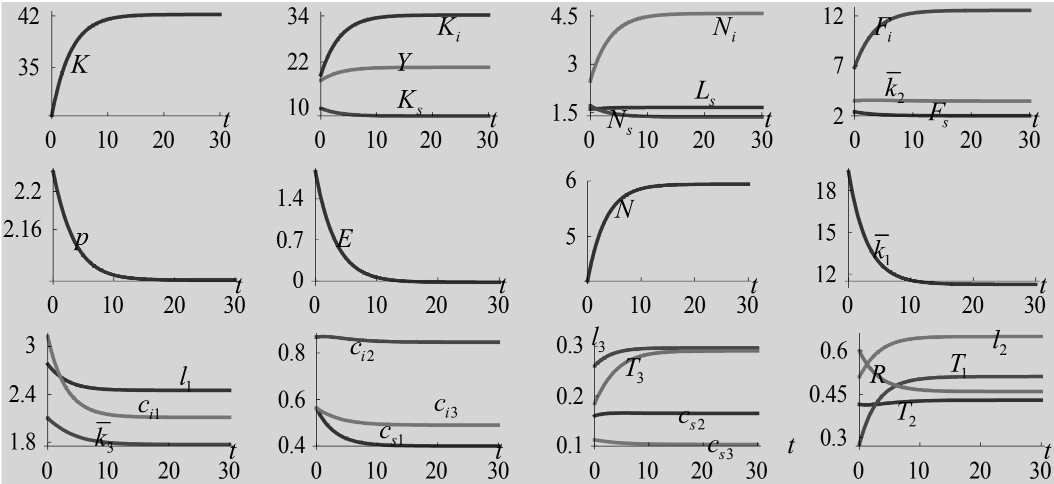
$$k_i = 7.48,\ k_s = 5.82,\ w_1 = 5.76,\ w_2 = 1.92,\ w_3 = 0.96.$$

The capital intensity of the industrial sector is higher than that of the service sector. Group $1's$ wage rate is highest among the three groups because of its highest human capital level. The initial conditions are specified as

$$R(0) = 0.6,\ \bar{k}_2(0) = 3.5,\ \bar{k}_3(0) = 2.1.$$

The motion of the economic system is plotted in Figure 1.

Figure 1: The Motion of the Economic System



In Figure 1, $Y(t)$ stands for the national product, defined as

$$Y(t) \equiv F_i(t) + p(t)F_s(t) + \sum_{j=1}^{J} l_j(t)\bar{N}_j .$$

The simulation confirms that the dynamic system achieves a stationary state by $t = 30$. From Figure 1 we see that the initial values of the land rent and price of services is fixed higher than their long-term equilibrium values. The two variables fall over time. The output level of the service sector falls. The output of the industrial good rises over time. The national product rises over time. The three groups all augment their work hours and the total labor force is increased. Over time the economy uses increasingly more foreign capital. The three groups reduce their consumption levels of the industrial good. Group $1's$ and Group $3's$ wealth are reduced, while Group $2's$ wealth level is slightly affected. Group $1's$ lot size is reduced, while Group $2's$ and Group $3's$ wealth level are increased. We confirmed that the system achieves at a stationary state in the long term. Simulation finds the following equilibrium values of the variables

$$Y = 20.64,\ K = 42.19,\ \bar{K} = 41.83,\ N = 5.94,\ E = -0.02,\ R = 0.46,\ p = 2.11,$$
$$F_s = 2.02,\ F_{\mathrm{i}} = 12.55,\ N_{\mathrm{i}} = 4.58,\ N_{\mathrm{s}} = 1.37,\ K_{\mathrm{i}} = 34.23,\ K_{\mathrm{s}} = 7.96,\ L_s = 1.67,$$
$$\bar{k}_1 = 11.26,\ \bar{k}_2 = 3.49,\ \bar{k}_3 = 1.77,\ c_{i1} = 1,\ c_{i2} = 0.4,\ c_{i3} = 0.23,\ c_{s1} = 0.4,$$
$$c_{s2} = 0.17,\ c_{s3} = 0.1,\ l_1 = 2.45,\ l_2 = 0.65,\ l_3 = 0.30,\ T_1 = 0.51,\ T_2 = 0.43,$$
$$T_3 = 0.29.$$

We also calculate the three eigenvalues as follows

$$\{-0.42, -0.38, -0.33\}.$$

Hence, the equilibrium point is stable. The existence of a unique stable equilibrium point is important as we can effectively conduct comparative dynamic analysis.

4. Comparative Dynamic Analysis

We plotted the motion of the economic system in the previous section. This section conducts comparative dynamic analysis, demonstrating how a change in a parameter alternates paths of the economic growth. As we can describe the motion of the system for any set of parameters, it is straightforward to make comparative dynamic analysis. This study uses the variable, $\bar{\Delta}x(t)$, to represent the change rate of the variable, $x(t)$, in percentage due to changes in the parameter value.

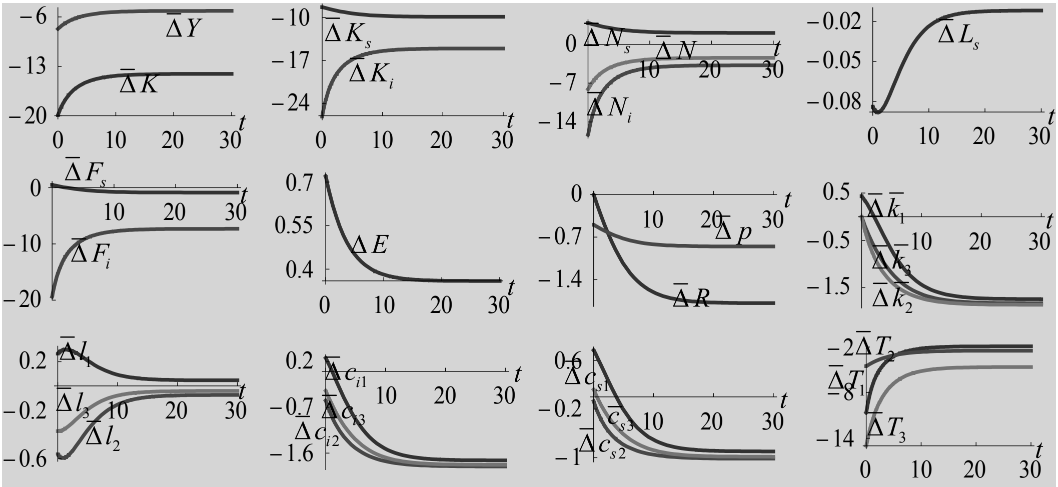
4.1 A Rise in the International Rate of Interest

First, we examine what will happen to the motion of the economic variables if the rate of interest is changed as follows: $r^* = 0.06 \Rightarrow 0.07$, where "$\Rightarrow$" stands for "being changed to". As the cost of capital in global markets is increased, the capital intensities of the two sectors and wage rates of the three groups are affected as follows

$$\bar{\Delta}k_i = \bar{\Delta}k_s = -11.69, \ \bar{\Delta}w_1 = \bar{\Delta}w_2 = \bar{\Delta}w_3 = -3.66.$$

The impacts on the time-dependent variables are plotted in Figure 2. As the wage rates are reduced, the work time of each group is reduced. The total supply of labor force and national output are decreased. As the cost of capital becomes more expensive, the economy utilizes less capital. Each sector uses less capital. The industrial sector's labor input is reduced, but the service sector's labor input is augmented. The service sector also employs less land. The price of service and land rent are reduced. The two sectors' output levels are lowered. The economy employs less capital socks and owns less wealth. The net result raises the return from net asset. Group $1's$ lot size is increased. The group's consumption levels of industrial goods and services are augmented initially but reduced subsequently. Group $2's$ and Group $3's$ lot sizes and consumption levels of industrial goods and services are reduced.

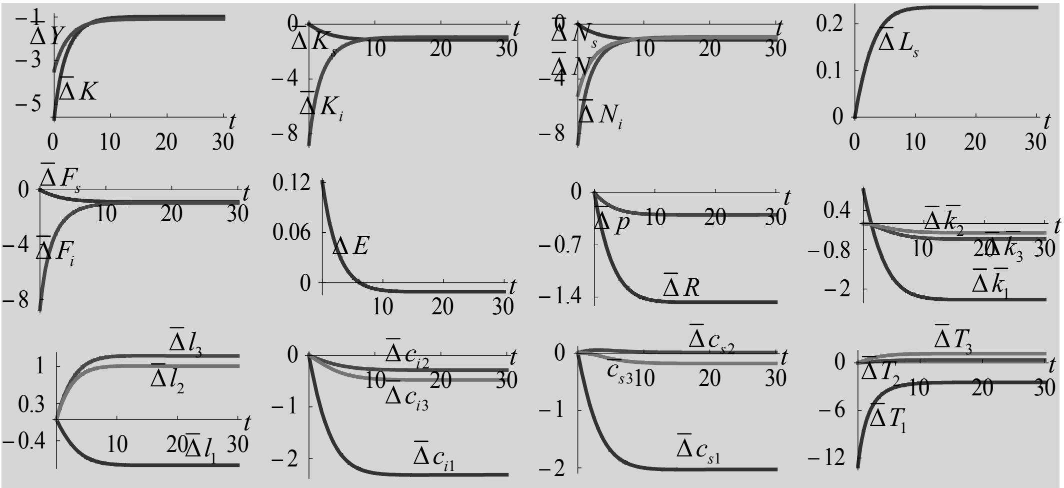
Figure 2: A Rise in the Rate of Interest in International Markets



4.2 A Rise in Group 1's Propensity to Stay at Home

Different preferences of different households are important for analyzing economic equilibrium and structure in the Walrasian general equilibrium theory. Nevertheless, the Walrasian theory does not contain proper economic mechanisms for analyzing effects of changes in one type of households on national economic growth as well as wealth and income distribution among different households. As our analytical framework integrates the economic mechanism of the Walrasian general equilibrium theory and neoclassical growth theory, in principle we can analyze effects a change in the preference of any people on the dynamic path of the economic growth. We now allow Group 1's propensity to stay at home to be increased as follows: $\sigma_{01} = 0.2 \Rightarrow 0.21.$ As the group appreciates more the time staying at home, Group 1's typical household stays home longer. The change in the propensity to stay has no impact on the capital intensities and wage rates, $\overline{\Delta} k_i = \overline{\Delta} k_s = \overline{\Delta} w_j = 0.$ The national labor force is reduced. The two sectors' labor inputs are reduced. As the capital intensities are fixed by international markets, the falling in the labor inputs also implies that the two sectors' capital intensities are reduced. The total capital employed by the economy is reduced. The service sector uses more land. The two sectors' output levels are lowered. As the output levels are reduced and the price and land rent are reduced, the national output falls. As the household from Group 1 works less hours, the per capita consumption levels of service and industrial goods are reduced. As the land rent falls, the household's income from land also falls. The households from Group 2 and Group 3 work more hours as their incomes fall. The two groups have larger lot sizes, even though they consume less services and industrial goods.

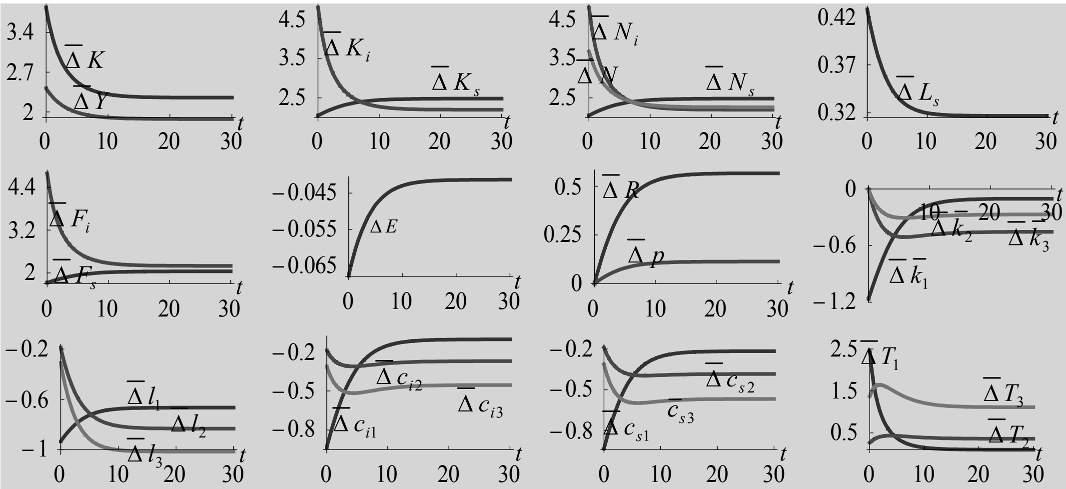
Figure 3: A Rise in Group 1's Propensity to Stay at Home



4.3 Group 3's Population Being Increased

It has been observed that the effect of population growth varies with the level of economic development and can be positive for some developed economies. Theoretical models with human capital predict situation-dependent interactions between population and economic growth (see, Ehrlich and Lui, 1997; Galor and Weil, 1999; Boucekkine, et al., 2002; Bretschger, 2013). There are also mixed conclusions in empirical studies on the issue (e.g., Furuoka, 2009; Yao et al., 2013). Our model also allows us to examine how each group's population may affect growth and inequality. We now allow group $3's$ population to be increase as follows: $N_3 : 5 \Rightarrow 5.2$. We have $\bar{\Delta} k_i = \bar{\Delta} k_s = \bar{\Delta} w_j = 0$. The rise in the population reduces all the groups' lot sizes and increases the land rent. The three groups also work longer hours. The three groups' consumption levels of service and industrial levels are all reduced. The total labor force, labor and capital inputs and output levels of the two sectors are all increased. We see that the national economic performance is increased, while the households suffer from the rise in the population.

Figure 4: Group 3's Population Being Increased



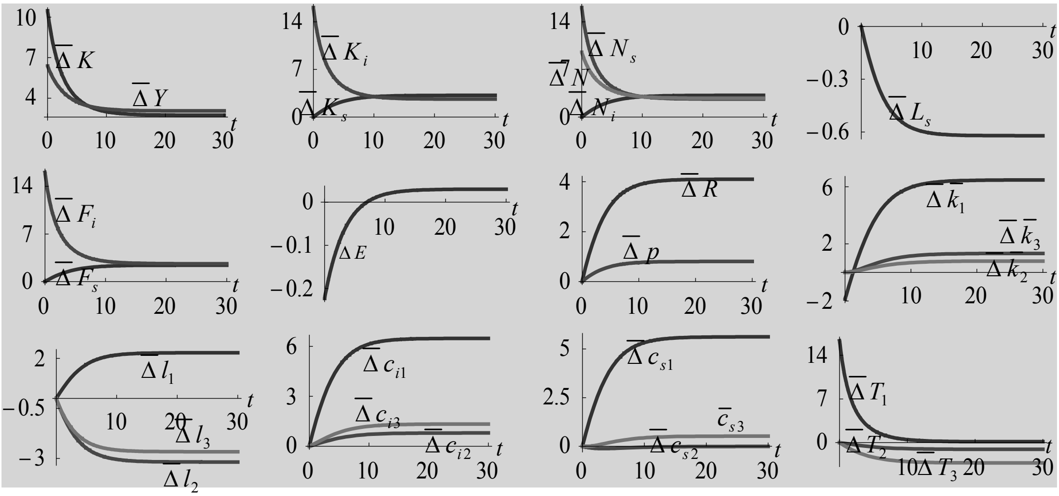
4.4 Group 1 Augmenting Human Capital

How changes in human capital can affect economic growth and inequality has been a main topic in modern economic theory and empirical research. Before the 1950s, as observed by Easterlin (1981), there were few people who had any formal education, outside North-Western Europe and North America. It has been argued by many researchers that human capital is an important factor for economic growth (e.g., Hanushek and Kimko, 2000; Barro, 2001; Krueger and Lindahl, 2001; Dimitra, et al. 2011; Castelló-Climent and Hidalgo-Cabrillana, 2012). Possible relations between human capital accumulation and earnings has caused great attention in empirical economics since Mincer (1974) published the seminal work in 1974. Earlier studies (e.g., Tilak, 1989) conclude that inequality is negatively related to spread education within countries. Could et al. (2001) find that the primary source of inequality growth within uneducated workers is due to increasing randomness, but inequality growth within educated workers is mainly due to changes in the composition and return to ability (see also Tselios, 2008; Fleisher et al. 2011). We now study how all the economic variables are connected to a change in Group $1's$ human capital during transitory processes and in long-term steady state. We now allow Group 1 to improve its human capital as follows: $h_1 : 3 \Rightarrow 3.2$. The effects on the capital intensities and wage rates are given as follows

$$\overline{\Delta} w_1 = 6.67, \ \overline{\Delta} k_i = \overline{\Delta} k_s = \overline{\Delta} w_2 = \overline{\Delta} w_3 = 0.$$

Group $1's$ wage rate is increased, while the other two groups' wage rates and capital intensities are not affected. As the group's wage rate is increased, the opportunity cost of staying at home becomes higher and Group 1 works longer hours. The total labor supply is increased. The two sectors' labor and capital inputs and output levels are all increased. The price of service and land rent are increased. The national output rises. As Group 1 demands more housing consumption, the group's lot size is increased, while the land input of the service sector and the other two groups' lot sizes are reduced. Group 1's wealth falls initially and rises subsequently. The other two groups' wealth levels are increased. The net result reduces initially and then raises the return from net asset. Group 1 and Group 3 consume more service and industrial good. Group 1's consumption of services and industrial good are slightly affected.

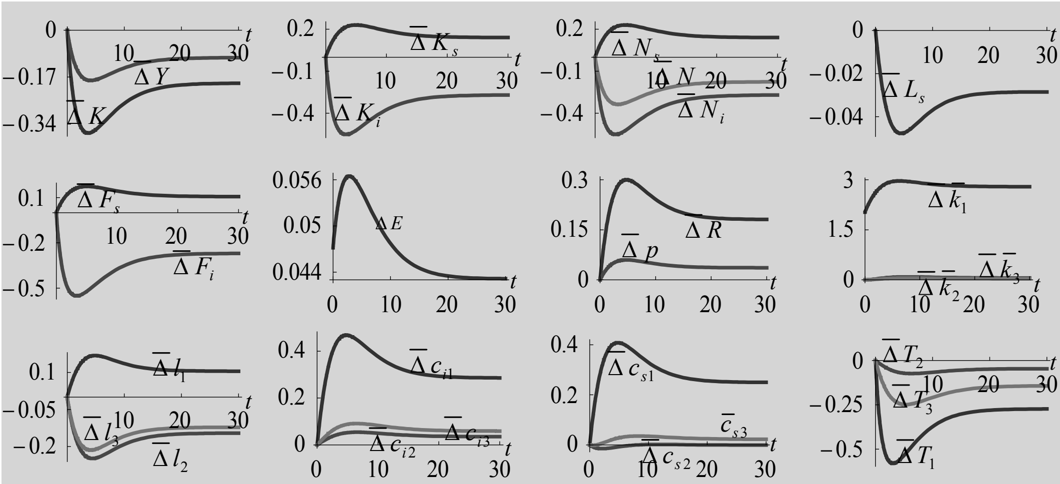
Figure 5: Group 1 Augmenting Human Capital



4.5 Group 1 Augmenting the Propensity to Save

First, we examine the case that Group 1 increases its propensity to save in the following way: $\lambda_{01} : 0.8 \Rightarrow 0.82$. The simulation results are given in Figure 6. Group 1's per capita wealth is increased. The capital intensities and wage rates are not affected. That is, $\overline{\Delta} k_i = \overline{\Delta} k_s = \overline{\Delta} w_j = 0$. We see that as Group 1 increases the propensity to save, the household from the group consumes more service and industrial good and has larger lot size. The household also reduces work hours. As the three groups all reduce work hours, the total labor supply falls. The labor input of the service sector is increased in association with the rise in the price of service. The labor input employed by the industrial sector is reduced. The total capital and capital employed by the industrial sectors are reduced, while the capital stock employed by the service sector is increased. The output level of the service sector rises, while the output level of the industrial sector falls. The return from net asset is slightly affected. It can be seen that the inequality between Group 1 and the other two groups are enlarged.

Figure 6: Group 1 Augmenting the Propensity to Save



5. Conclusion

This paper is concerned with the relationship between growth and inequality in a two-sector growth modeling framework. Both extensive theoretical and empirical studies find ambiguous relationships between growth and inequality. This paper proposed an economic growth model of a small open economy with fixed resource (land) in a perfectly competitive economy. The production side consists of one service sector and one industrial sector. Following the traditional literature of small open growth economies, we treat the rate of interest fixed in international market. We used an alternative utility function proposed by Zhang to describe the behavior of households. In our approach the wealth and income inequality is due to heterogeneity in households' preferences and human capital levels as well as the households' initial wealth. We first built a model for any number of types of household. Then we gave a computation procedure for simulating model with proper initial conditions. For illustration we simulated the model for the economy with three types of household. The system has a unique stable equilibrium point for the given parameters. We simulated the motion of the national economy and carried out comparative dynamic analysis with regard to changes in the rate of interest, the population, the propensity to stay at home, and the propensity to save. The comparative dynamic analysis provides some important insights. For instance, as the rich group increases its propensity to save, not only the group's per capita wealth is increased, but also the group's consumption levels of service and industrial good and lot size are increased. All the households of the three groups reduce work hours. The labor input of the service sector is increased in association with the rise in price of service. The labor input employed by the industrial sector is reduced. The total capital and capital employed by the industrial sectors are reduced, while the capital stock employed by the service sector is increased. The output level of the service sector rises, while the output level of the industrial sector falls. The inequality between the rich group and the other two groups are enlarged.

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Appendix

Proving the Lemma

We determined k_i, w, and k_s as functions of r^* and A_i. From $K_j = k_j N_j$ and (9), we have

$$K_i = \left(K - k_s N\right) k_v k_i, \quad K_s = \left(k_i N - K\right) k_v k_s, \tag{A1}$$

where we omit time variable in expressions. From (5), we solve

$$R = \frac{w_s N_s}{L_s}, \tag{A2}$$

where we also use $l_s = L_s / N_s$ and $w_s \equiv w \gamma_s / \beta_s$. Inserting (A2) in (20) implies

$$\sum_{j=1}^{J} l_j \bar{N}_j + \frac{w_s N_s}{R} = L. \tag{A3}$$

From the definition of $\bar{y}_j$, we get

$$\bar{y}_j = \left(1 + r^*\right) \bar{k}_j + w_j T_0 + \frac{R L}{\bar{N}}. \tag{A4}$$

Equation (A4) and $l_j = \eta_j \bar{y}_j / R$ in (13) implies

$$l_j = \frac{\left(1 + r^*\right) \eta_j \bar{k}_j + w_j \eta_j T_0}{R} + \frac{\eta_j L}{\bar{N}}. \tag{A5}$$

Inserting (A5) in (A3) implies

$$\sum_{j=1}^{J} \bar{n}_j \bar{k}_j + w_s N_s = \eta_0 R - \bar{\eta}_0, \tag{A6}$$

where

$$\bar{n}_j \equiv \left(1 + r^*\right) \eta_j \bar{N}_j, \quad \bar{\eta}_0 \equiv T_0 \sum_{j=1}^{J} w_j \eta_j \bar{N}_j, \quad \eta_0 \equiv \left(1 - \frac{1}{\bar{N}} \sum_{j=1}^{J} \eta_j \bar{N}_j\right) L.$$

From $r^* + \delta_k = \alpha_s p F_s / K_s$ and (16) we have

$$\sum_{j=1}^{J} c_{sj} \bar{N}_j = \frac{\left(r^* + \delta_k\right) K_s}{\alpha_s p}. \tag{A7}$$

Inserting $c_{sj} = \gamma_j \bar{y}_j / p$ in (A7) implies

$$\sum_{j=1}^{J} \gamma_j \bar{y}_j \bar{N}_j = \frac{\left(r^* + \delta_k\right) K_s}{\alpha_s}. \tag{A8}$$

Insert (A4) into (A8)

$$\sum_{j=1}^{J} \bar{\gamma}_j \bar{k}_j + \bar{\gamma}_0 = \frac{\left(r^* + \delta_k\right) K_s}{\alpha_s}, \tag{A9}$$

where we also use (10) and

$$\bar{\gamma}_j \equiv \left(1 + r^*\right) \gamma_j \bar{N}_j , \ \ \bar{\gamma}_0 \equiv \sum_{j=1}^{J} \left(w_j \gamma_j \bar{N}_j T_0 + \frac{R L}{\bar{N}} \gamma_j \bar{N}_j \right).$$

We assume $\varepsilon \neq 1$. From (6) we have

$$p = p_0 R^{\gamma_s} , \tag{A10}$$

where we also use $l_s = w_s / R$ from (A2) and

$$p_0 \equiv \frac{w}{\beta_s A_s k_s^{\alpha_s} w_s^{\gamma_s}}.$$

Insert (A10) in (A9)

$$\sum_{j=1}^{J} \bar{\gamma}_j \bar{k}_j + \bar{\gamma}_0 + a p_0^{1-\varepsilon} y_f^{\varphi} R^{\gamma_s (1-\varepsilon)} = \frac{\left(r^* + \delta_k\right) K_s}{\alpha_s}. \tag{A11}$$

Substitute $N_s = \left(k_i N - K\right) k_v$ from (9) into (A6) and $K_s = \left(k_i N - K\right) k_v k_s$ from (A1) into (A11) respectively yields

$$\sum_{j=1}^{J} \bar{n}_j \bar{k}_j + \left(k_i N - K\right) w_s k_v = \eta_0 R - \bar{\eta}_0 ,$$

$$\sum_{j=1}^{J} \bar{\gamma}_j \bar{k}_j + \bar{\gamma}_0 = \left(k_i N - K\right) \bar{k}_v , \tag{A12}$$

where

$$\bar{k}_v \equiv \left(\frac{r^* + \delta_k}{\alpha_s} \right) k_v k_s .$$

From (A12), we solve

$$\bar{k}_1 = \Omega\left(R, \{\bar{k}_j\}\right), \tag{A13}$$

where

$$\Omega\left(R, \{\bar{k}_j\}\right) \equiv \frac{\left(\eta_0 R - \bar{\eta}_0\right)\bar{k}_v - w_s \bar{\gamma}_0 k_v - \sum_{j=2}^{J}\left(\bar{k}_v \bar{n}_j + k_v w_s \bar{\gamma}_j\right)\bar{k}_j}{\bar{k}_v \bar{n}_1 + k_v w_s \bar{\gamma}_1}.$$

From (18), we have

$$T_j = T_0 - \frac{\sigma_j \bar{y}_j}{w_j}. \tag{A14}$$

Inserting (A4) in (A14) implies

$$T_j = \left(1 - \sigma_j\right)T_0 - \left(1 + r^*\right)\frac{\sigma_j}{w_j}\bar{k}_j - \frac{\sigma_j L}{w_j \bar{N}} R. \tag{A15}$$

From (1) and (A15), we have

$$N = \tilde{N} - \sum_{j=1}^{J} \bar{\sigma}_j \bar{k}_j - \tilde{\sigma} R, \tag{A16}$$

where

$$\tilde{N} \equiv T_0 \sum_{j=1}^{J}\left(1 - \sigma_j\right)h_j \bar{N}_j, \ \ \bar{\sigma}_j = \left(1 + r^*\right)\frac{h_j \bar{N}_j \sigma_j}{w_j}, \ \ \tilde{\sigma} \equiv \frac{L}{\bar{N}}\sum_{j=1}^{J}\frac{\sigma_j h_j \bar{N}_j}{w_j}.$$

The following procedure shows how to find all the variables as functions of R and $\{\bar{k}_j\}$: $\bar{k}_1$ by (A13) → $\bar{y}$ by (A4) → p by (A10) → T_j by (A15) → T_{hj}, l_j, c_{ij}, c_{sj}, s_j by (18) → N by (A16) → K by (A12) → K_i and K_s by (A1) → N_i and N_s by (9) → $\bar{K}(t)$ by (22) → L_s by (A2) → F_i by (2) → F_s by (5). From this procedure and (19), we have

$$\dot{\bar{k}}_1 = \Omega_0\left(R, \{\bar{k}_j\}\right) \equiv s_1 - \bar{k}_1, \tag{A17}$$

$$\dot{\bar{k}}_j = \Omega_j\left(R, \{\bar{k}_j\}\right) \equiv s_j - \bar{k}_j, \ \ j = 2, ..., J. \tag{A18}$$

Taking derivatives of (A13) with respect to time implies

$$\dot{\bar{k}}_1 = \frac{\partial \Omega}{\partial R}\dot{R} + \sum_{j=2}^{J}\Omega_j \frac{\partial \Omega}{\partial \bar{k}_j}, \tag{A19}$$

where we use (A18). We do not provide the expression of the partial derivatives because they are tedious. Equaling the right-hand sides of (A17) and (A19), we get

$$\dot{R} = \Omega_1\left(R, \left\{\bar{k}_j\right\}\right) \equiv \left(\Omega_0 - \sum_{j=2}^{J} \Omega_j \frac{\partial \Omega}{\partial \bar{k}_j}\right)\left(\frac{\partial \Omega}{\partial R}\right)^{-1}. \tag{A20}$$

We thus proved Lemma.